There exist two options for the asymptotic behaviour of the function near the vertical asymptote:

$$\text{if} \quad E_V < E_i, \tag{3.36a}$$

$$\text{then} \quad \begin{cases} \lim\limits_{E \to E_{V-}} \varphi_2(E) & = -\infty \\ \lim\limits_{E \to E_{V+}} \varphi_2(E) & = +\infty; \end{cases} \tag{3.36b}$$

$$\text{if} \quad E_V > E_i, \tag{3.37a}$$

$$\text{then} \quad \begin{cases} \lim\limits_{E \to E_{V-}} \varphi_2(E) & = +\infty \\ \lim\limits_{E \to E_{V+}} \varphi_2(E) & = -\infty. \end{cases} \tag{3.37b}$$

The A-intercept and the horizontal asymptote are easily found:

$$\varphi_2(0) = -\frac{\varepsilon}{u_1} . \tag{3.38}$$

$$\lim_{E \to +\infty} \varphi_2(E) = -\frac{w}{v} . \tag{3.39}$$

Let us study the intervals of the increase and decrease of the function

$$\varphi_2^1(E) = \frac{u_1 e \varepsilon}{\left(vE - \frac{u_1 e}{e+E}\right)^2} \left[x^2 + \frac{w}{\varepsilon}\, x - \frac{v}{u_1 e}\right], \tag{3.40}$$

where

$$x = \frac{1}{e+E} . \tag{3.41}$$

$\varphi_2^1(E)$ changes its sign from $(-)$ to $(+)$ when x passes in the positive direction through the value

$$x_+ = -\frac{w}{2\varepsilon} + \sqrt{\frac{w}{2\varepsilon} + \frac{v}{u_1 e}} . \tag{3.42}$$

That means $\varphi_2^1(E)$ changes its sign from (+) to (−) when E exceeds

$$E_+ = \frac{1}{x_+} - e. \tag{3.43}$$

Hence $\varphi_2(E)$ increases whenever $E < E_+$ (except for E_V if $E_V < E_+$), reaches its maximum at $E = E_+$, and then decreases for all $E > E_+$ (of course, except $E = E_V$ if $E_V > E_+$). It is impossible to say without additional assumptions whether E_+ is positive or negative.

Thus, the graph of $\varphi_2(E)$ is represented by one of the following options depicted in Figure 3.2. As one can observe (Fig. 3.2a), there is one and only one intersection of the graphs, i.e. one and only one equilibrium, when $E_V < E_i$.

Let us show that conditions (3.30), which are our intuitive assumptions for the model parameters, lead exactly to this case $(E_V < E_i)$.

$$\begin{aligned} E_V &= -\frac{e}{2} + \frac{e}{2}\sqrt{1 + \frac{u_1}{4ev}} \\ &= -\frac{e}{2} + \frac{e}{2}\left[1 + \frac{u_1}{8ev} + O\Big(\big(\frac{u_1}{4ev}\big)^2\Big)\right] \\ &= \frac{u_1}{16v} + O\Big(\big(\frac{u_1}{4ev}\big)^2\Big). \end{aligned}$$

From (3.30) it follows that $\frac{\varepsilon}{w} \sim 1$ and $\frac{u_1}{16v} \sim \frac{1}{16}$. Hence, $\frac{\varepsilon}{w} > \frac{u_1}{16v}$. Small corrections of the order of $(\frac{u_1}{v})^2 \cdot \frac{1}{16e^2} \ll 1$ cannot alter that inequality. Thus $E_V < E_i$.

If (3.37) holds, there may be no equilibria (Fig. 3.2c), one (Fig. 3.2b) or more equilibria (Fig. 3.2d). The number of equilibria in this case $(E_i < E_V)$ depends upon the comparable magnitudes of ue/α and E_i as well as on the concavity properties of $\varphi_2(E)$. As system (3.31) may be reduced to a cubic equation with respect to E, the number of equilibria cannot exceed three.

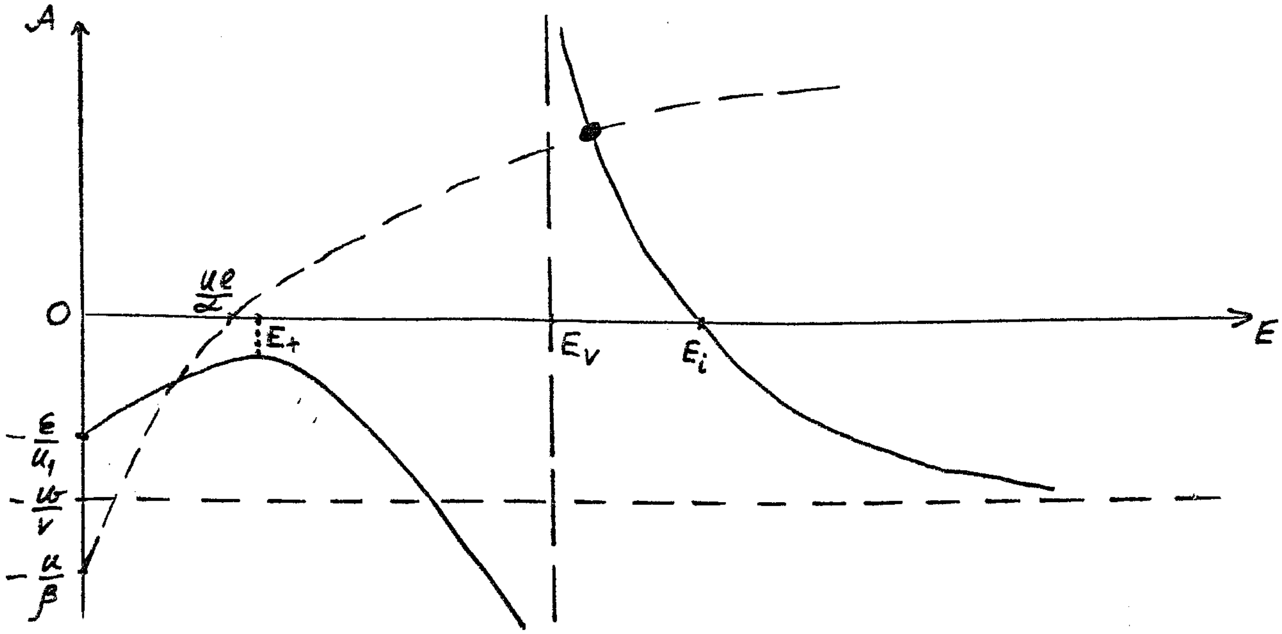

Figure 3.2 (a)

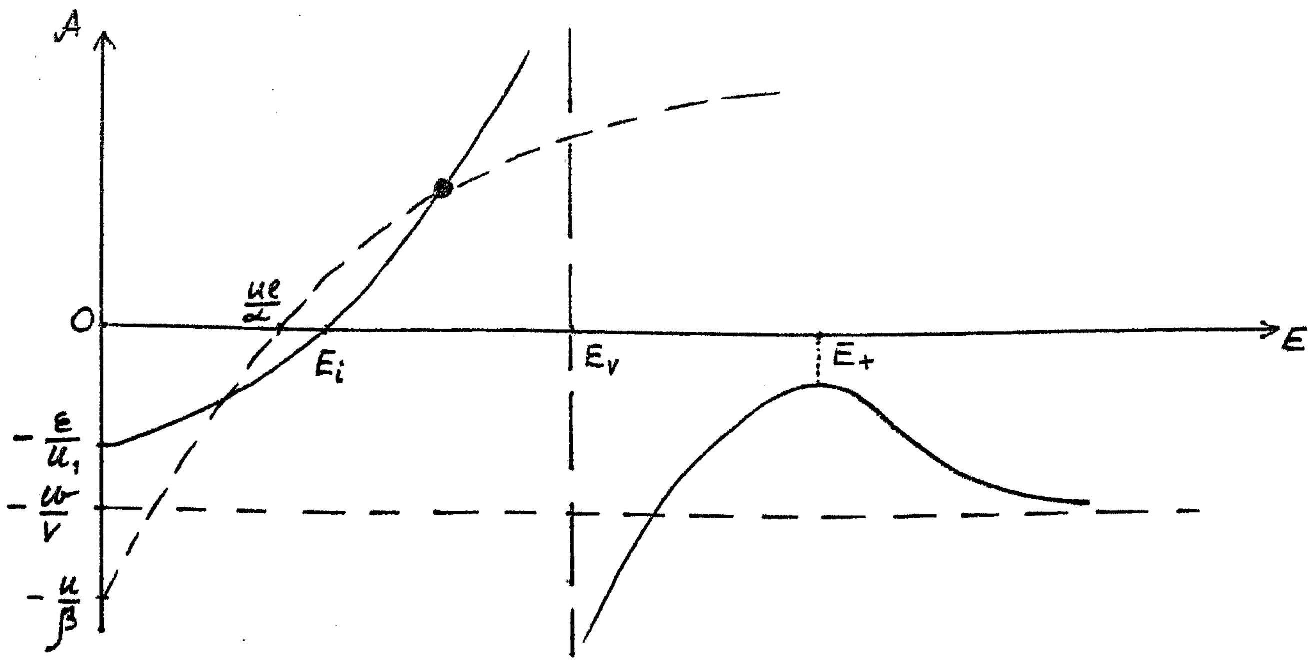

Figure 3.2 (b)

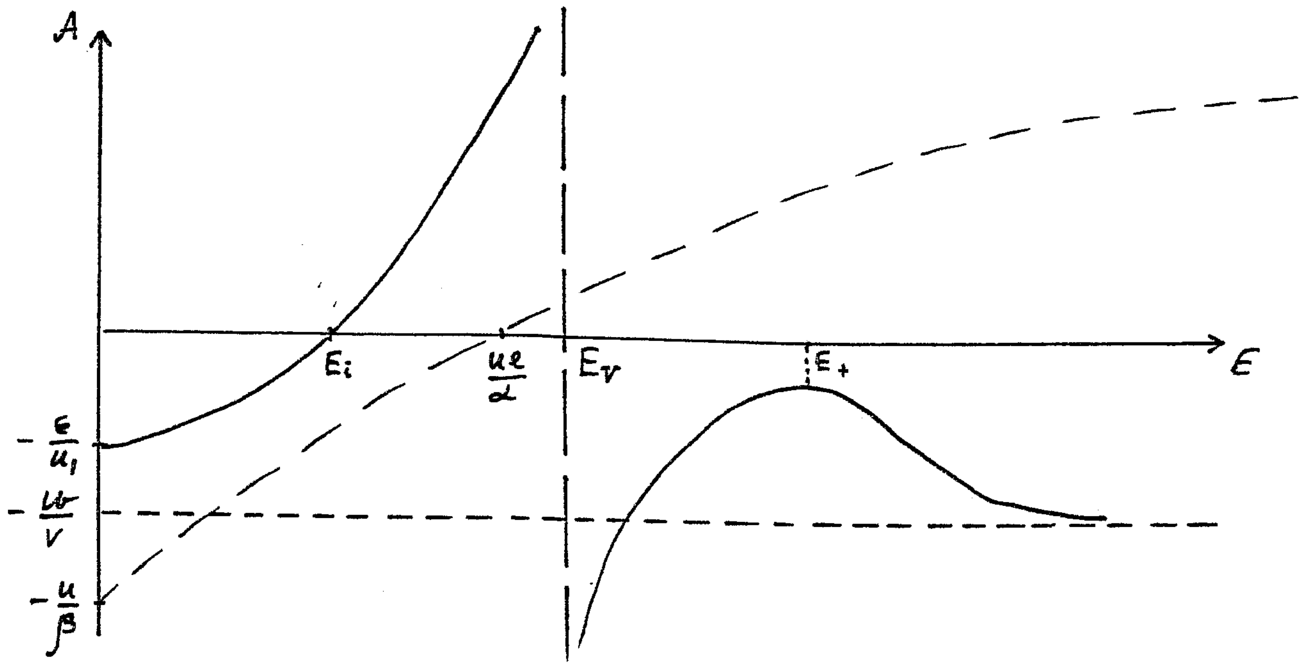

Figure 3.2 (c)

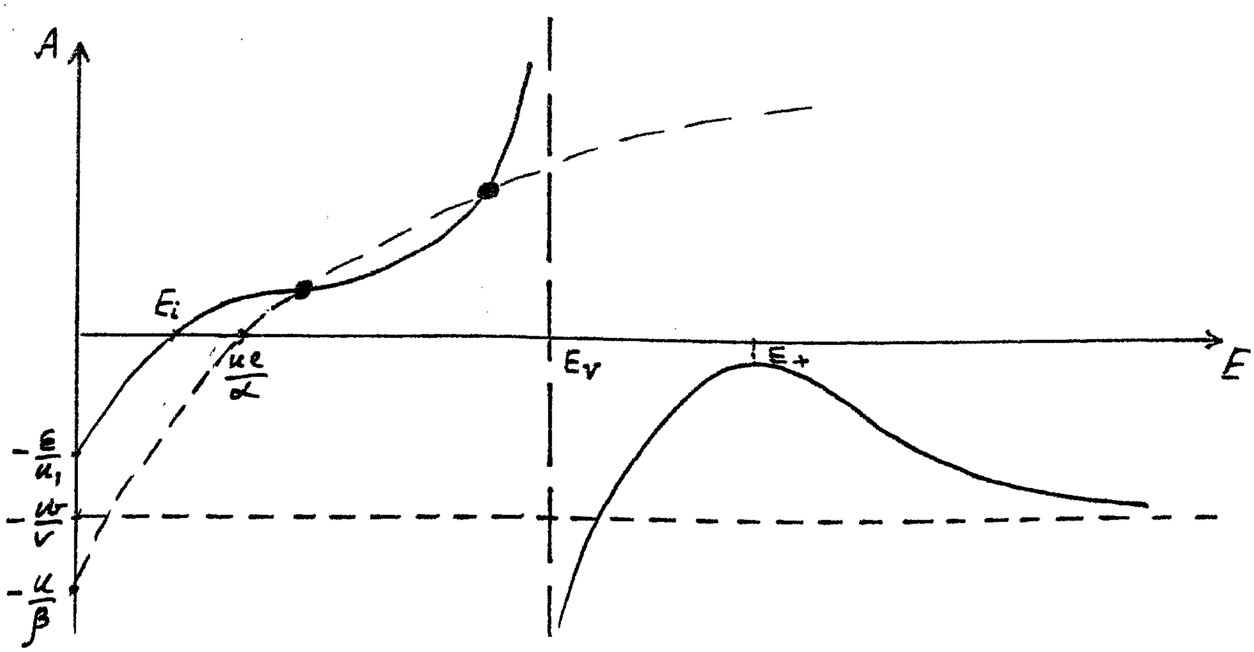

Figure 3.2 (d)

The variational matrix of this system in the plane $I = 0$ reduces to

$$\widehat{V} = \begin{pmatrix} -\mu\beta\widehat{A} & \mu\Big(\frac{\gamma\widehat{E}}{e+\widehat{E}} - \delta\Big)\frac{\widehat{A}}{b(a+\widehat{A})} & \frac{\mu e\widehat{A}(\alpha+u)}{(e+\widehat{E})^2} \\ 0 & -\xi + \frac{\delta}{b}\frac{\widehat{A}}{a+\widehat{A}} & 0 \\ -v\widehat{E} + \frac{u_1 e}{e+\widehat{E}} & 0 & -w - v\widehat{A} - \frac{u_1 e\widehat{A}}{e+\widehat{E}} \end{pmatrix}. \tag{3.44}$$

The eigenvalue corresponding to the I-direction equals

$$\lambda_2 = -\xi + \frac{\delta b}{}\frac{\widehat{A}}{a+\widehat{A}} < -\xi + \frac{\delta}{b}. \tag{3.45}$$

Then if

$$\delta > b\xi, \tag{3.46}$$

λ_2 is positive, and the equilibrium $\widehat{F}(\widehat{A}, 0, \widehat{E})$ is a repeller in the I-direction.

□

Summary of the Stability Properties for the Boundary Sets

If $\varepsilon > 0$, the only axial equilibrium is $F^*(0, 0, \frac{\varepsilon}{w})$, unstable locally in the A-direction under the condition (3.2.3).

In the $I - E$ plane the only equilibrium is $\widehat{F}(0, \frac{-\xi+\sqrt{\xi^2+4\eta\zeta}}{2\eta}, \frac{\varepsilon}{w})$. It is unstable locally in the A-direction if (3.28) is satisfied.

There are no equilibria in the $A - I$ plane. We note here that in this plane

$$\frac{dE}{dt}\Big|_{E=0} = u_1 A > 0, \tag{3.47}$$

signifying that the plane is a repeller. There are, under the assumptions of Proposition 3.2, also equilibria in the A, E-plane. They are repellers locally in the I-direction.

We shall complete this subsection by showing that there are no periodic orbits in the positive quadrants of the $I - E$ and $A - E$ planes. We need this property for establishing the uniform persistence of our system.

In the $I-E$ plane the system (2.11) reduces to

$$\frac{dI}{dt} = \zeta - \xi I - \eta I^2 \tag{3.48a}$$

$$\frac{dE}{dt} = \varepsilon - wE. \tag{3.48b}$$

The divergence of the flow

$$\mathcal{D}(I,E) = \frac{\partial(\zeta - \xi I - \eta I^2)}{\partial I} + \frac{\partial(\varepsilon - wE)}{\partial E} = -\xi - 2\eta I - w \tag{3.49}$$

is always negative, which implies, by Bendixson's Criterion [9, p. 245] the absence of closed orbits in the positive $I-E$ quadrant.

The vector flow in the positive $A-E$ plane is defined by

$$\frac{dA}{dt} = \mu\left[\frac{\alpha E - ue}{e+E}\, A - \beta A^2\right], \tag{3.50a}$$

$$\frac{dE}{dt} = \varepsilon - wE - vAE + u_1\,\frac{Ae}{e+E}\,. \tag{3.50b}$$

Choosing $B(A,E) = \frac{1}{A}$ and computing the divergence of the vector field (3.50) multiplied by $B(A,E)$, we get

$$\begin{aligned}\mathcal{D}(A,E) &= \frac{\partial}{\partial A}\,(A^{-1}\dot{A}) + \frac{\partial}{\partial E}\,(A^{-1}\dot{E}) \\ &= -\mu\beta - \frac{w}{A} - v - \frac{u_1 e}{(e+E)^2} < 0 \end{aligned} \tag{3.51}$$

for $(A,E) \in \text{int } R_+^2$. Hence, by Dulac's Criterion [9, p. 246] there is no closed orbit lying in the positive quadrant of the $A-E$ plane.

b) Persistence

This term is borrowed from the theory of mathematical ecology where several species coexist within a common closed ecosystem and implies the actual survival of all populations included in the system. For, our system representing a rural economy consisting of 3 components – agriculture, industry and the ecosphere, persistence is interpreted to mean that none of these components vanish.

Various levels of persistence may be observed [6]. *Weak persistence* means that positive solutions do not asymptotically approach the boundary, i.e. the solution set where at least one of the components vanishes. *Persistence* means that each solution with strictly positive initial conditions is eventually at some positive distance from the boundary.

Neither weak persistence nor persistence guarantee the survival of the system; they simply ensure that extinction is not deterministic. Small stochastic perturbation of the system may result in the solution being driven to the boundary.

Uniform persistence is a more robust property: it ensures that strictly positive solutions are uniformly distanced from the boundary set. In formal terms there exists such a value $\chi_0 > 0$, that for all points $\big(A(t), I(t), E(t)\big)$ along a trajectory of the system

$$\begin{gathered}\lim_{t\to\infty} \inf A(t) > \chi_0; \quad \lim_{t\to\infty} \inf I(t) > \chi_0; \\ \lim_{t\to\infty} \inf E(t) > \chi_0.\end{gathered} \tag{3.52}$$

As follows from the previous subsection and Theorem 2.1 by Freedman and Waltman [7], system (2.11) exhibits persistence.

Moreover, the absence of periodic trajectories in the $I - E$ and $A - E$ planes, together with the existence of only one locally axial equilibrium unstable in the A-direction and the absence of any equilibria in the $A - I$ plane, ensures acyclicity of the boundary set. The latter together with dissipativity of the system and isolatedness of the boundary set leads to uniform persistence and the existence of an interior equilibrium of the system [8].

Specifically, the following proposition and its corollary are valid.

Proposition 3.3. *Let* $\varepsilon > 0$ *and the following inequalities be satisfied:*

$$\frac{\alpha\widehat{E} - ue}{e + \widehat{E}} + \left(\gamma \frac{\widehat{E}}{e + \widehat{E}} - \delta\right) \frac{\widehat{I}}{a(b + \widehat{I})} > 0 \tag{3.28}$$

with $\widehat{I} = \frac{-\xi + \sqrt{\xi^2 + 4\eta\zeta}}{2\eta}$, $\widehat{E} = \frac{\varepsilon}{w}$;

$$\frac{\alpha\varepsilon}{w} - ue > 0; \tag{3.23}$$

$$\delta > b\xi. \tag{3.46}$$

Then, under the hypotheses of Proposition 3.2, *system* (2.11) *exhibits uniform persistence.*

Corollary 3.4. *Under the hypotheses of Proposition* 3.3 *there exists a positive interior equilibrium for system* (2.11).

These results are of the greatest importance. First, they show that a rural economic system represented by our model is sustainable. Second the hypotheses of Proposition 3.3 provide us with the sufficient conditions for sustainability.

c) Stabilization of the ecosphere.

It has been proven in the preceding subsection that dynamical system (2.11) possesses an interior equilibrium in the positive octant.

Now we confirm this result constructively. We show how investments from the agricultural system to rehabilitate the ecological wealth of the whole system stabilize the E-variable at a particular level. The investments result in interior equilibria located near this level.

The conditions defining the location of an interior equilibrium for system (2.11) are

$$\frac{\alpha E - ue}{e + E} - \beta A + \left(\gamma \frac{E}{e + E} - \delta\right) \frac{I}{(a + A)(b + I)} = 0 \tag{3.53a}$$

$$\zeta - \xi I - \eta I^2 + \delta \frac{AI}{(a+A)(b+I)} = 0 \tag{3.53b}$$

$$\varepsilon - wE - vAE + u_1 A \frac{e}{e+E} = 0. \tag{3.53c}$$

It is evident from (3.53c) that in this special case of the cessation of agricultural activities in the system $(A = 0)$, ecological wealth would stabilize at the level

$$E = \frac{\varepsilon}{w} \equiv \widetilde{E}. \tag{3.54}$$

Hereafter we shall call this level the 'natural' level of ecospheric wealth. The term 'natural' may be interpreted in terms such as the long term biological equilibrium level of soil organic matter, associated with natural un-grazed grasslands or mature forests.

Let us impose achievement of this 'natural' level of ecospheric wealth as a necessary condition on an active agricultural system with $(A > 0)$. We may now demonstrate that a management strategy that meets this condition ensures sustainability of the whole A, E, I system. Furthermore, such a conforming strategy provides the basis for further system development leading to the growth of agricultural wealth in the long run.

After the stabilization requirement (3.54) is imposed, (3.53c) reduces to

$$-v\widetilde{E} + u_1 \frac{e}{e+\widetilde{E}} = 0. \tag{3.55}$$

The latter condition may be used to determine the magnitude of u_1, the coefficient representing the level of investment needed to keep the ecosphere stable:

$$u_1 = \frac{v}{e} \frac{e+\widetilde{E}}{\widetilde{E}} \equiv \widetilde{u}_1. \tag{3.56}$$

Now, when $E = \widetilde{E} =$ const., system (2.11) reduces to the following two-dimensional system:

$$\frac{dA}{dt} = \mu \left[\frac{\alpha\widetilde{E} - ue}{e+\widetilde{E}} A - \beta A^2 + \left(\gamma \frac{\widetilde{E}}{e+\widetilde{E}} - \delta \right) \frac{AI}{(a+A)(b+I)} \right] \tag{3.57a}$$

$$\frac{dI}{dt} = \zeta - \xi I - \eta I^2 + \delta \, \frac{AI}{(a+A)(b+I)} \, . \tag{3.57b}$$

Denoting

$$\widetilde{\alpha} \equiv \frac{\alpha \widetilde{E} - ue}{e + \widetilde{E}} \tag{3.58}$$

$$\text{and} \quad \widetilde{\gamma} \equiv \gamma \, \frac{\widetilde{E}}{e + \widetilde{E}} \, , \tag{3.59}$$

we can rewrite our system (3.57) as

$$\frac{dA}{dt} = \mu \Big[\widetilde{\alpha} A - \beta A^2 + (\widetilde{\gamma} - \delta) \, \frac{AI}{(a+A)(b+I)} \Big] \tag{3.60a}$$

$$\frac{dI}{dt} = \zeta - \xi I - \eta I^2 + \delta \, \frac{AI}{(a+A)(b+I)} \, . \tag{3.60b}$$

With the exception of parameter ζ, the latter system is identical to the two-dimensional model considered in [1]. It can be shown both analytically by means of the implicit function theorem and numerically, that the introduction of a small ζ does not affect qualitatively and almost does not affect quantitatively the dynamics of the system.

Thus we can rely on the results of paper [1], keeping in mind the ecological content of newly defined parameters $\widetilde{\alpha}$ and $\widetilde{\gamma}$.

More specifically, it has been shown in [1] that system (3.60) does have at least one interior equilibrium. Furthermore a series of *bifurcations*, qualitative changes in the structure of the system dynamics, may be described. Before considering these bifurcations and their implications for management of the relationship between agriculture and the environment in Section IV, we discuss possible consequences of destabilization of the ecosphere.

d) Effects of destabilization of the ecosphere

First, we should notice that small deviations of the value of u from the "naturally stabilizing" $\widetilde{u}$ defined through $\widetilde{u}_1$ given by (3.56) as

$$\widetilde{u} = \frac{u_1}{\mu}, \tag{3.61}$$

do not affect the system behaviour. They result in small damping or a sustained E-value with small amplitude oscillation near $\widetilde{E}$. As for the A- and I-variables, these deviations do not affect the corresponding outcomes. This shows that the model is robust with respect to the small fluctuations always taking place in real life economic systems.

Insufficient investment, in the ecosphere, defined as $u < \widetilde{u}$ may result in some undesirable outcomes for a given system. For example, as we can observe on Figure 3.3, agricultural and industrial wealth of a system with $u = 1$ stabilize at some high level equilibrium (Fig. 3.3a), whereas for the same system with $u = 0.5$, the magnitudes of A and I oscillate near substantially lower values.

We should notice here that sensitivity of the system dynamics to the changes of u is much greater when some other parameters of the system are close to their bifurcation values. For instance, the system represented in Figures 3.3a and b, is characterized by $e = 5$. This value of e is slightly lower than the bifurcation value of e when $u = 1$ and slightly higher than the bifurcation value of e when $u = 0.5$. The latter means that the system with $e = 5$, $u = 1$ has already undergone a saddle-node bifurcation which resulted in the creation of a locally stable equilibrium at a comparatively high level of A and I. This equilibrium serves as the global attractor of the system.

When $u = 0.5$, the bifurcation value for e is lower than 5, therefore, the system with $u = 0.5$, $e = 5$ has only one equilibrium, a stable focus located at low levels of A and I. This equilibrium attracts all initial conditions.

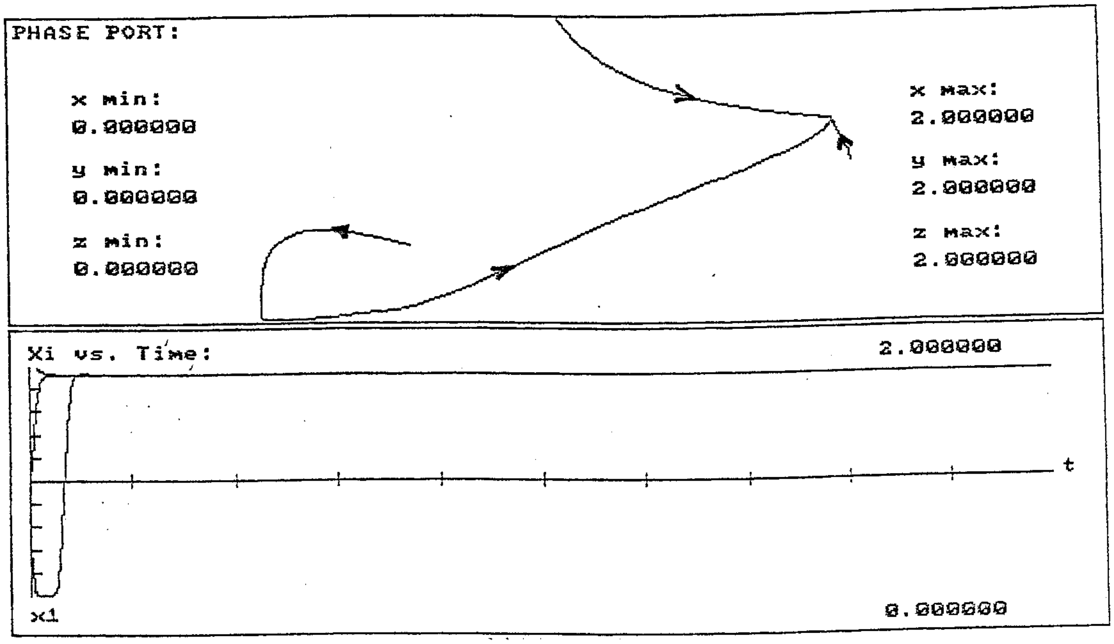

Figure 3.3a $e = 5$; $u = 1$

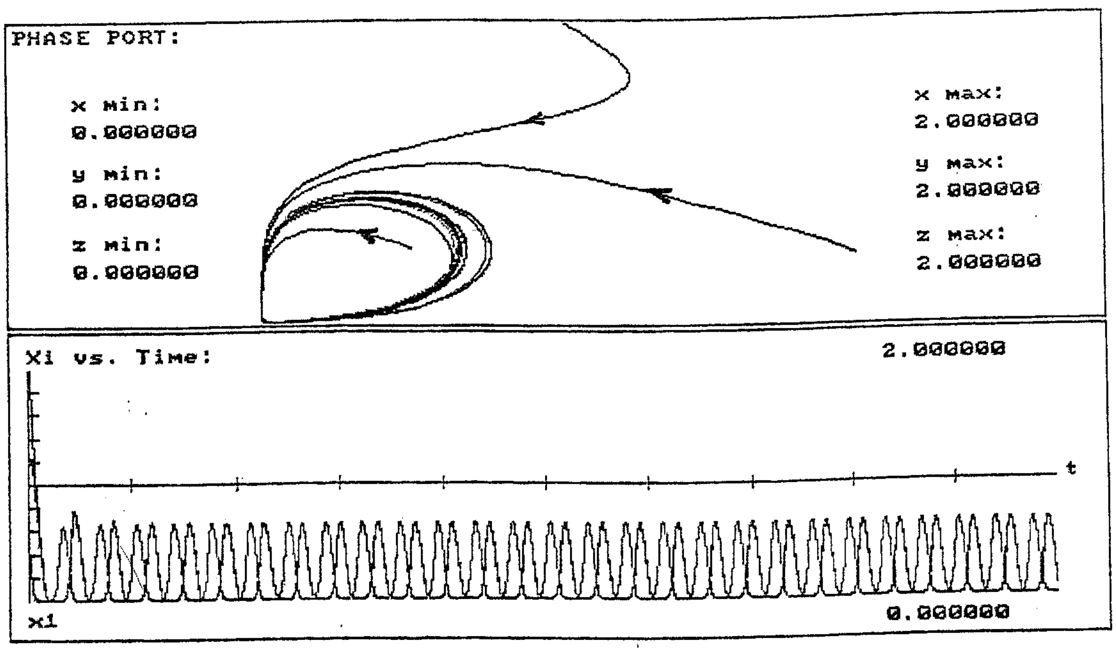

Figure 3.3b $e = 5$; $u = 0.5$

For the same system with $e = 6$, corresponding to the slower recovery of the ecosphere, the difference between the outcomes for the cases of $u = 1$ and $u = 0.5$ is only quantitative, not qualitative (see Fig. 3.4). In both cases values of A and I oscillate near a low level equilibrium, though the amplitude of the oscillation is larger in the case of greater investments in the ecosphere (Fig. 3.4a).

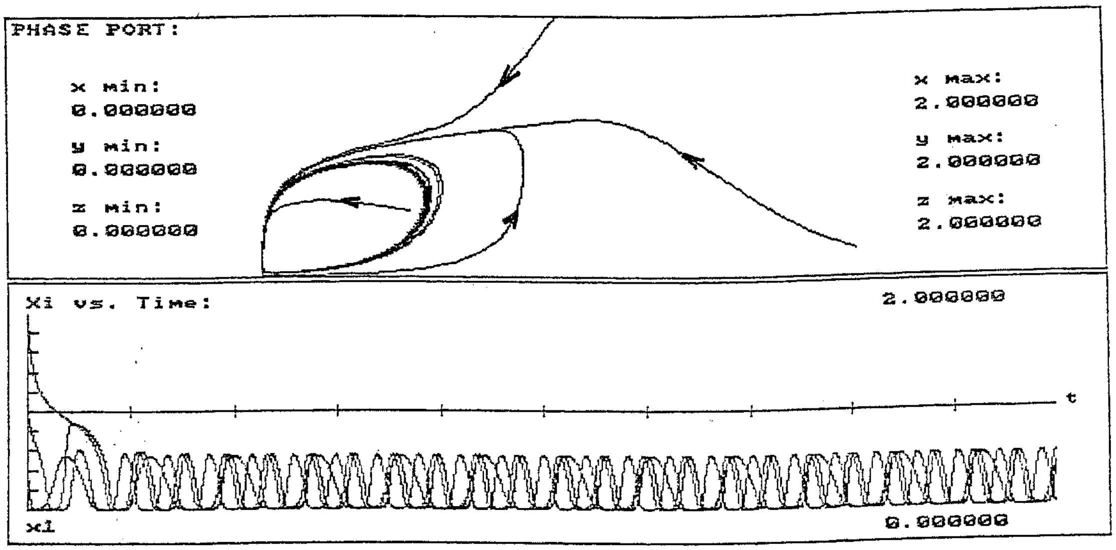

Figure 3.4a $e = 6$; $u = 1$

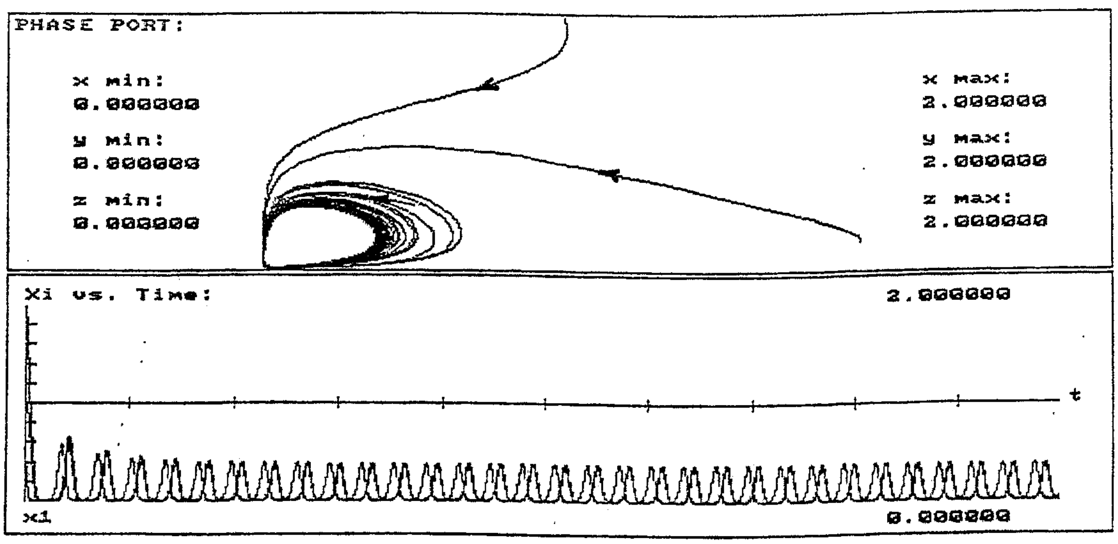

Figure 3.4b $e = 6$; $u = 0.5$

We can conclude that investments to rehabilitate the environment can be used most successfully in systems with a sufficiently high recovery rate for the ecosphere.

The danger of excessive investments in E, represented by $u > \widetilde{u}$, lies in the possible violation of the conditions of uniform persistence. It is likely to happen if the value of α determined by the carrying capacity of the environment is not high enough. Particularly, conditions (3.28) and (3.23) may be broken. The latter makes equilibria $\widehat{F}$ and F^* lying in the plane $A = 0$ local attractors. Consequently it is possible that some initial conditions will end up at zero level of A.

Yet for very favorable initial conditions, very high values of u may lead to the creation of new attractors with desirable outcomes (Fig. 3.5). Thus, if a system has acquired substantial wealth, it may achieve even better results by means of generous investments supporting the ecosphere. In spite of this possible advantage of choosing a large u strategy, keeping the ecospheric wealth near its *natural* level not only comes at lower cost, but is a lower risk policy. Throughout the balance of this paper we consider E to be stabilized or oscillating with a small amplitude about its natural level $E = \widetilde{E}$.

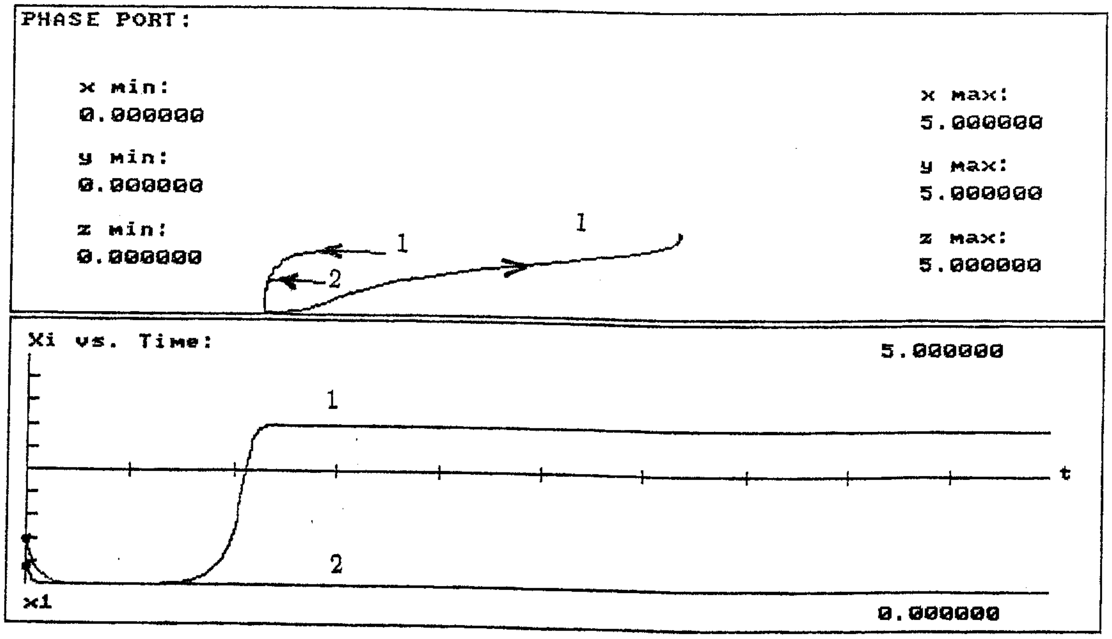

Figure 3.5. Trajectory starting at favorable initial condition (1) ends up at the equilibrium with high value of A. Unfavorablle initial condition (2) results in "extinction" of the agricultural component.

IV. Escape from the Low Level Economic Trap

1. Structural and Control Parameters

So far we have used mostly analytic techniques to determine the dynamic properties of our model. In our further discussion numerical computer experiments play an important role. The first task is to choose appropriate ranges for the values of the *structural* parameters of the system. This choice should be *representative*, in the sense of ensuring the possibility of observing the whole spectrum of system behaviour.

Our choice of values, hereafter called the *basic set*, is the following:

$$\begin{aligned} &\alpha = 20; \quad \beta = 1; \quad \gamma = 1.5; \quad \delta = 2; \quad a = b = 0.25; \\ &\zeta = 0; \quad \xi = 1; \quad \eta = 0.1; \quad v = 1; \\ &w = 0.1; \quad \varepsilon = 0.1. \end{aligned} \tag{4.1}$$

Parameter $u_1 = \mu u$ is chosen either in accordance with (3.56) or does not differ from $\widetilde{u}_1$ by more than 15%. (Therefore, the ecosphere is stabilized at $E = \widetilde{E} = 1$.)

The rest of the parameters, particularly

$$\mu, \; e, \tag{4.2}$$

are designated as *control* parameters. Control parameters may be used as management instruments to achieve desirable outcomes, by making the system undergo a sequence of bifurcations. This approach to management of outcomes is strategic and structural, heavily focused on nurturing a strong learning culture. In certain situations some of the parameters included in (4.1) may be also used as control parameters.

2. Bifurcations

Bifurcation is a structural change in system dynamics occurring when the magnitude of some parameter passes through a certain number, called the bifurcation value. For example, a new equilibrium may be created, or the one that already exists, changes its stability properties. Bifurcation may create or destroy a periodic trajectory. A *strange attractor*, an aperiodic attractor exhibiting high sensitivity to initial conditions, may emerge or disappear as a result of a bifurcation.

Consider four types of structural change our system may undergo in the presence of bifurcation.

(a) When the a and b parameters change from 1 to 0.25 (economic recovery is getting faster), the only attractor of the system, a stable node, located at a low level of A and I transforms into a stable focus and moves to a higher level of I (Figure 4.1). For all the figures graphing the results of numerical experiments presented in the paper, x_1 (or x) stands for A, x_2 (or y) for I and x_3 (or z) for E, if not otherwise specified.

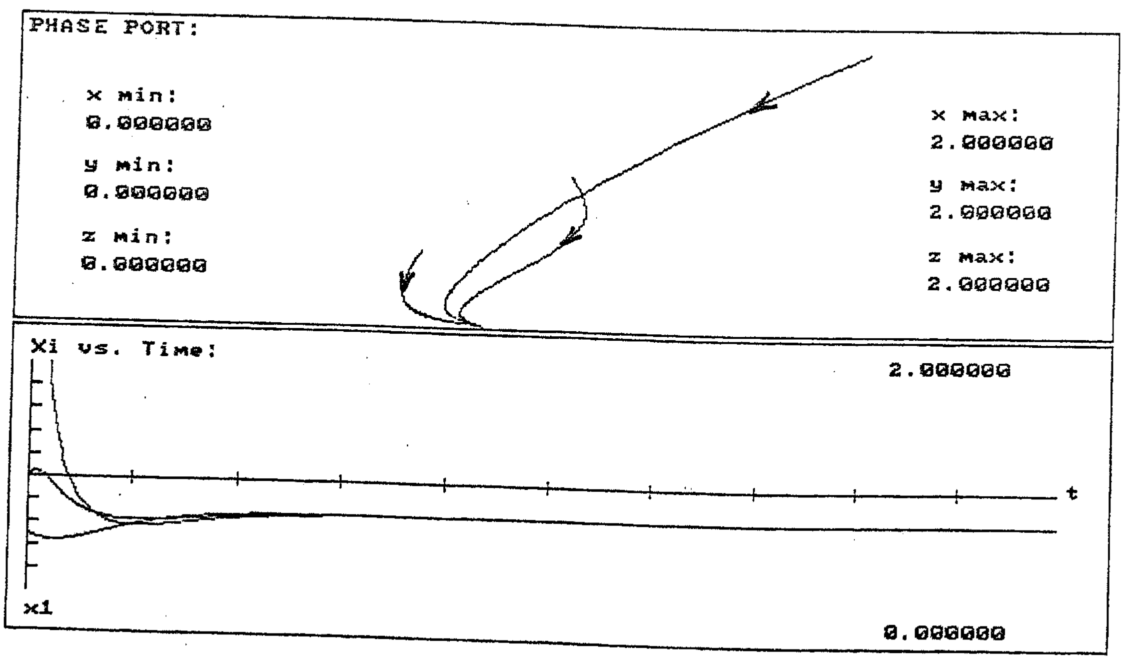

Figure 4.1a $a = 1$; $u = 1.1$; $e = 6$; $\mu = 0.5$

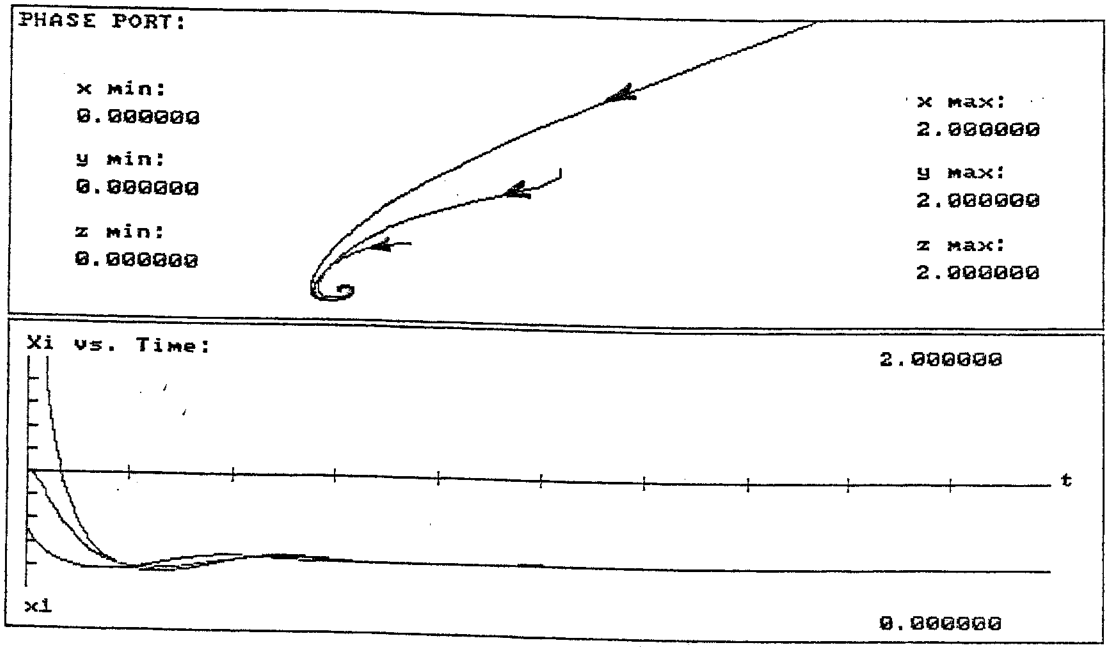

Figure 4.1b $a = 0.5$; $u = 1.1$; $e = 6$; $\mu = 0.5$

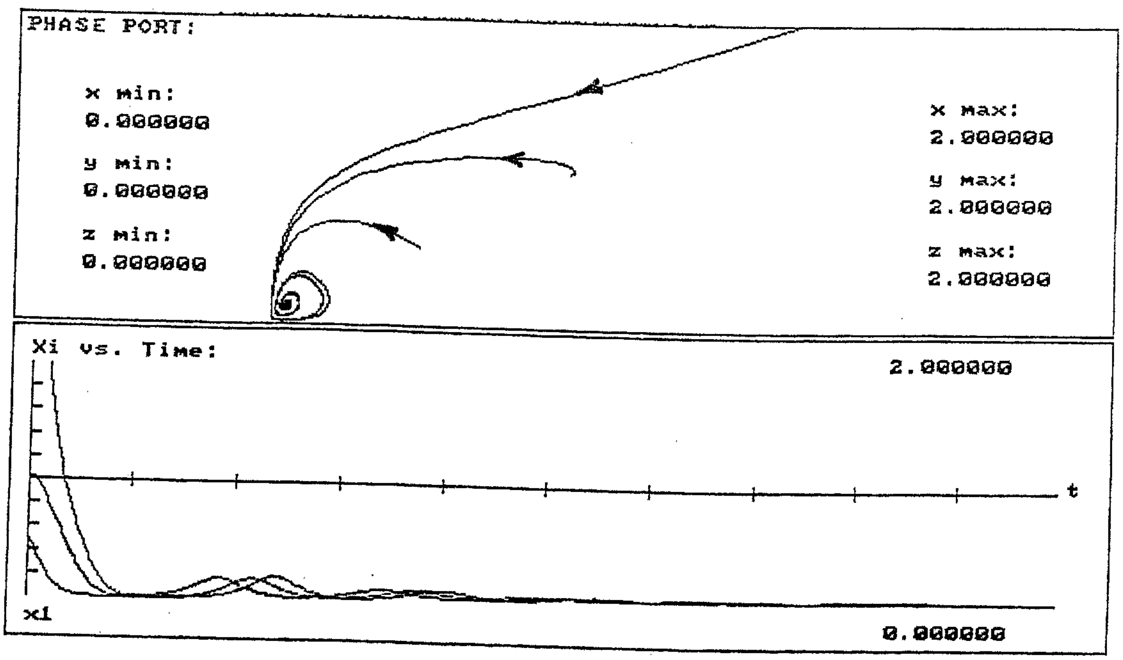

Figure 4.1c $a = 0.25$; $u = 1.1$; $e = 6$; $\mu = 0.5$